Finsler Manifolds with Nonpositive Flag Curvature and Constant S-curvature

Zhongmin Shen

Abstract

The flag curvature is a natural extension of the sectional curvature in Riemannian geometry, and the S-curvature is a non-Riemannian quantity which vanishes for Riemannian metrics. There are (incomplete) non-Riemannian Finsler metrics on an open subset in R^n with negative flag curvature and constant S-curvature. In this paper, we are going to show a global rigidity theorem that every Finsler metric with negative flag curvature and constant S-curvature must be Riemannian if the manifold is compact. We also study the nonpositive flag curvature case.

1 Introduction

One of important problems in Finsler geometry is to understand the geometric meanings of various quantities and their impacts on the global geometric structures. Imaging a Finsler manifold as an Easter egg and a Riemannian manifold as a white egg, Finsler manifolds are not only curved, but also very "colorful". The flag curvature $\mathbf{K}$ tells us how curved is the Finsler manifold at a point. There are several non-Riemannian quantities which describe the "color" and its rate of change over the manifold, such as the mean Cartan torsion $\mathbf{I}$, the mean Landsberg curvature $\mathbf{J}$ and the S-curvature $\mathbf{S}$ (see [17] or Section 2 below). These quantities interact with the flag curvature in a delicate way. The mean Landsberg curvature and the S-curvature reveal different non-Riemannian properties. For examples, there is a family of Finsler metrics on S^3 with $\mathbf{K}=1$ and $\mathbf{S}=0$ [5]. However, every local Finsler metric with $\mathbf{K}=1$ and $\mathbf{J}=0$ must be Riemannian (Theorem 9.1.1 in [17]).

An n-dimensional Finsler metric is said to have *constant S-curvature* if $\mathbf{S}=(n+1)cF$ for some constant c. It is known that every Randers metric of constant flag curvature has constant S-curvature [3], [4]. This is one of our motivations to consider Finsler metrics of constant S-curvature. In this paper, we are going to prove the following global metric rigidity theorem.

Theorem 1.1 *Let (M,F) be an n-dimensional compact boundaryless Finsler manifold with constant S-curvature, i.e., $\mathbf{S}=(n+1)cF$ for some constant c.*

(a) If F has negative flag curvature, $\mathbf{K}<0$, then it must be Riemannian;

(b) *If F has nonpositive flag curvature, $\mathbf{K} \leq 0$, then the mean Landsberg curvature vanishes, $\mathbf{J} = 0$, and the flag curvature $\mathbf{K}(P, y) = 0$ for the flags $P = \mathrm{span}\{y, \mathbf{I}_y\} \subset T_x M$ whenever $\mathbf{I}_y \neq 0$.*

The compactness in Theorem 1.1 (a) can not be dropped. Consider the following family of Finsler metrics on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2} + \langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \tag{1}$$

where $y \in T_x\mathrm{B}^n \cong \mathrm{R}^n$ and $a \in \mathrm{R}^n$ is an arbitrary constant vector with $|a| < 1$. It is proved that F has constant flag curvature $\mathbf{K} = -\frac{1}{4}$ and constant S-curvature $\mathbf{S} = \frac{1}{2}(n+1)F$ (see [17][18]). Clearly, F is not Riemannian.

The compactness in Theorem 1.1 (b) can not be dropped. Let $n \geq 2$ and

$$\mathcal{U} := \Big\{ p = (s, t, \bar{p}) \in \mathrm{R}^2 \times \mathrm{R}^{n-2} \ \Big|\ s^2 + t^2 < 1 \Big\}.$$

Define

$$F := \frac{\sqrt{\Big(-tu + sv\Big)^2 + |y|^2\Big(1 - s^2 - t^2\Big)} - \Big(-tu + sv\Big)}{1 - s^2 - t^2},$$

where $y = (u, v, \bar{y}) \in T_p\mathcal{U} \cong \mathrm{R}^n$ and $p = (s, t, \bar{p}) \in \mathcal{U}$. F is an incomplete Finsler metric on Ω with $\mathbf{K} = 0$ and $\mathbf{S} = 0$, but $\mathbf{J} \neq 0$ [20].

The compactness condition in Theorem 1.1 can be replaced by a completeness condition together with certain growth condition on the mean Cartan torsion. See Theorems 4.1 and 4.2 below.

Corollary 1.2 *Let (M, F) be a compact boundaryless Berwald manifold with nonpositive flag curvature. Then the following hold,*

(a) *If F has negative flag curvature, $\mathbf{K} < 0$, then it must be Riemannian;*

(b) *If F has nonpositive flag curvature, $\mathbf{K} \leq 0$, then $\mathbf{K}(P, y) = 0$ for the flag $P = \mathrm{span}\{y, \mathbf{I}_y\}$ whenever $\mathbf{I}_y \neq 0$.*

In dimension two, we have the following

Corollary 1.3 *Let (M, F) be a compact boundaryless Finsler surface. Suppose that $\mathbf{K} \leq 0$ and $\mathbf{S} = 3cF$ for some constant c, then F is either locally Minkowskian or Riemannian.*

The proof is simple. First, by Theorem 4.1 below, we know that $\mathbf{J} = 0$, then the theorem follows from Theorem 7.3.2 in [2].

In dimension $n \geq 3$, we have some non-trivial examples satisfying the conditions and conclusions in Theorem 1.1 (b). Let (N, h) be an arbitrary closed hyperbolic Riemannian manifold. For any $\epsilon \geq 0$, let

$$F_\epsilon := \sqrt{h^2(\bar{x}, \bar{y}) + w^2 + \epsilon\sqrt{h^4(\bar{x}, \bar{y}) + w^4}},$$

where $x = (\bar{x}, s) \in M$ and $y = \bar{y} \oplus w\frac{\partial}{\partial s} \in T_xM$. This family of Finsler metrics is constructed by Z.I. Szabó in his classification of Berwald metrics [21]. It is known that each F_ϵ is a Berwald metric. Thus $\mathbf{J} = 0$ and $\mathbf{S} = 0$ [17]. Further it can be shown that F_ϵ satisfies that $\mathbf{K} \leq 0$ and $\mathbf{K}(P, y) = 0$ for $P = \mathrm{span}\{y, \mathbf{I}_y\}$. The proof will be given in Section 5 below. A natural problem arises: Is the Finsler metric in Theorem 1.1 (b) a Berwald metric? This problem remains open.

Finally, we should point out that there are already several global rigidity results on the metric structure of Finsler manifolds with $\mathbf{K} \leq 0$. For example, H. Akbar-Zadeh proves that every closed Finsler manifold with $\mathbf{K} = -1$ must be Riemannian and every closed Finsler manifold with $\mathbf{K} = 0$ must be locally Minkowskian [1]. Mo-Shen prove that every closed Finsler manifold of scalar curvature with $\mathbf{K} < 0$ must be of Randers type in dimension ≥ 3 [15]. Here a Finsler metric F is said to be *of scalar curvature* if the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is independent of P for any given direction $y \in T_xM$. Riemannian metrics of scalar curvature must have isotropic sectional curvature $\mathbf{K} = \mathbf{K}(x)$, hence they have constant sectional curvature in dimension $n \geq 3$ by the Schur Lemma. But there are lots of Finsler metrics of scalar curvature which have not been completely classified yet.

2 Preliminaries

In this section, we are going to give a brief description on the flag curvature and the above mentioned non-Riemannian quantities.

Let M be an n-dimensional manifold and let $\pi : TM_o := TM \setminus \{0\} \to M$ denote the slit tangent bundle. The pull-back tangent bundle is defined by $\pi^*TM := \{(x, y, v) \mid 0 \neq y, v \in T_xM\}$ and the pull-back cotangent bundle is defined by $\pi^*T^*M := \{\pi^*\theta \mid \theta \in T^*M\}$.

By definition, a Finsler metric F on a manifold M is a nonnegative function on TM which is positively y-homogeneous of degree one with positive definite fundamental tensor $\mathbf{g} := g_{ij}dx^i \otimes dx^j$ on π^*TM, where $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}(x, y)$. A special class of Finsler metrics are Randers metrics in the form $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$ for any $x \in M$.

For a Finsler metric F, the volume $dV = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathrm{R}^n \Big| \ F\Big(x,\ y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}. \tag{2}$$

When $F = \sqrt{g_{ij}(x)y^iy^j}$ is Riemannian, then $\sigma_F(x) = \sqrt{\det(g_{ij}(x))}$. In general, the following quantity is not equal to zero,

$$\tau(x, y) := \ln\Big[\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)}\Big].$$

$\tau = \tau(x, y)$ is a scalar function on TM_o, which is called the *distortion* [17]. The distortion is our primary non-Riemannian quantity. Let

$$I_i := \frac{\partial \tau}{\partial y^i}(x, y) = \frac{1}{2} g^{jk}(x, y) \frac{\partial g_{jk}}{\partial y^i}(x, y). \tag{3}$$

We have

$$I_i y^i = 0. \tag{4}$$

The tensor $\mathbf{I} := I_i dx^i$ on TM_o is called the *mean Cartan tensor*. According to Deicke's theorem [11], F is Riemannian if and only if $\mathbf{I} = 0$. Define the norm of $\mathbf{I}$ at a point $x \in M$ by

$$\|\mathbf{I}\|_x := \sup_{0 \neq y \in T_x M} \sqrt{I_i(x, y) g^{ij}(x, y) I_j(x, y)}.$$

For a point $p \in M$, let

$$\mathcal{I}_p(r) := \sup_{\min(d(p,x), d(x,p)) < r} \|\mathbf{I}\|_x.$$

The mean Cartan tensor $\mathbf{I}$ is said to grow sub-linearly if for any point $p \in M$,

$$\mathcal{I}_p(r) = o(r), \qquad (r \to +\infty).$$

$\mathbf{I}$ is said to grow *sub-exponentially* at rate of $k = 1$ if for any point $p \in M$,

$$\mathcal{I}_p(r) = o(e^r), \qquad (r \to +\infty).$$

It is known that for a Randers metric $F = \alpha + \beta$, $\mathbf{I}$ is bounded, i.e.,

$$\|\mathbf{I}\|_x \leq \frac{n+1}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\beta\|_x^2}} < \frac{n+1}{\sqrt{2}}, \qquad x \in M.$$

The bound in dimension two is suggested by B. Lackey. See Proposition 7.1.2 in [17] for a proof.

The geodesics in a Finsler manifold are characterized by a system of second order ordinary differential equations

$$\ddot{\sigma}^i + 2G^i(\sigma, \dot{\sigma}) = 0,$$

where $G^i = G^i(x, y)$ are positively y-homogeneous functions of degree two. When F is Riemannian, $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y \in T_x M$. A Finsler metric with such a property called a *Berwald metric*. There are many non-Riemannian Berwald manifolds (see Section 5 below).

For a non-zero vector $y \in T_x M$, set

$$\mathbf{S}(x, y) := \frac{d}{dt}\Big[\tau\Big(\sigma(t), \dot{\sigma}(t)\Big)\Big]|_{t=0},$$

where $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. $\mathbf{S} = \mathbf{S}(x, y)$ is a scalar function on TM_o which is called the *S-curvature* [16][17]. Let $dV =$

$\sigma_F(x)dx^1\cdots dx^n$ be the volume form on M. The S-curvature can be expressed by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m}(x,y) - y^m\frac{\partial}{\partial x^m}\Big[\ln\sigma_F(x)\Big]. \tag{5}$$

It is proved that $\mathbf{S}=0$ for Berwald metrics [16][17]. An n-dimensional Finsler metric F is said to *have constant S-curvature* if there is a constant c such that $\mathbf{S}=(n+1)cF$. It is known that all Randers metrics of constant flag curvature must have constant S-curvature [3] (see [4] for the classification of such metrics).

There is a distinguished linear connection ∇ on π^*TM which is called the *Chern connection* [10]. Let $\{\mathbf{e}_i\}$ be a local frame for π^*TM and $\{\omega^i\}$ the dual local frame for π^*T^*M. ∇ can be expressed by

$$\nabla V = \Big\{dV^i + V^j\omega_j^{\ i}\Big\}\otimes\mathbf{e}_i,$$

where $V=V^i\mathbf{e}_i\in C^\infty(\pi^*TM)$. The Chern connection can be viewed as a generalization of the Levi-Civita connection in Riemannian geometry. Let

$$\omega^{n+i} := dy^i + y^j\omega_j^{\ i},$$

where y^i are local functions on TM_o defined by the canonical section $\mathbf{Y}=y^i\mathbf{e}_i$ of π^*TM. We obtain a local coframe $\{\omega^i,\omega^{n+i}\}$ for $T^*(TM_o)$.

Let

$$\Omega^i := d\omega^{n+i} - \omega^{n+j}\wedge\omega_j^{\ i}.$$

Ω^i can be expressed as follows,

$$\Omega^i = \frac{1}{2}R^i_{\ kl}\omega^k\wedge\omega^l - L^i_{\ kl}\omega^k\wedge\omega^{n+l},$$

where $R^i_{\ kl}+R^i_{\ lk}=0$ and $L^i_{\ kl}=L^i_{\ lk}$. The anti-symmetric tensor $\mathbf{R}=R^i_{\ kl}\mathbf{e}_i\otimes\omega^k\otimes\omega^l$ is called the *Riemann tensor* and the symmetric tensor $\mathbf{L}=L^i_{\ kl}\mathbf{e}_i\otimes\omega^k\otimes\omega^l$ is called the Landsberg tensor.

Let

$$R^i_{\ k} := R^i_{\ kl}y^l, \qquad R_{jk} := g_{ij}R^i_{\ k}.$$

We have

$$R^i_{\ k}y^k = 0, \qquad R_{jk} = R_{kj}. \tag{6}$$

See [17] for details. The tensor $\mathbf{R}:=R^i_{\ k}\mathbf{e}_i\otimes\omega^k$ is still called the *Riemann tensor*. The notion of Riemann (curvature) tensor for general Finsler metrics is introduced by L. Berwald using the Berwald connection [6][7]. Let

$$J_k := L^m_{\ km}.$$

The tensor $\mathbf{J}:=J_i\omega^i$ is called the *mean Landsberg tensor*. For a Berwald metric, $\mathbf{J}=0$ [17].

For a scalar function on TM_o, say τ, we define its covariant derivatives by

$$d\tau = \tau_{|k}\omega^k + \tau_{\cdot k}\omega^{n+k}.$$

From (3), we have

$$\tau_{\cdot i} = \frac{\partial \tau}{\partial y^i} = I_i.$$

We have

$$\mathbf{S} := \tau_{|m} y^m.$$

For a tensor, say, $\mathbf{I} = I_i\omega^i$, the covariant derivatives are defined in a canonical way by

$$dI_i - I_k {\omega_i}^k = I_{i|k}\omega^k + I_{i\cdot k}\omega^{n+k}.$$

We have

$$J_i = I_{i|m} y^m \tag{7}$$

Hence

$$J_i y^i = 0. \tag{8}$$

See [17] for details.

Now we interpret the above geometric quantities in a different way.

Let F be a Finsler metric on an n-dimensional manifold M. For a non-zero tangent vector $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, define

$$\mathbf{g}_y(u, v) := g_{ij}(x, y)u^i v^j, \qquad u = u^i \frac{\partial}{\partial x^i}|_x, v = v^j \frac{\partial}{\partial x^j}|_x \in T_xM,$$

where $g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}(x, y)$. Each $\mathbf{g}_y$ is an inner product on the tangent space T_xM.

The Riemann tensor can be viewed as a family of endomorphisms on tangent spaces.

$$\mathbf{R}_y(u) := {R^i}_k(x, y)u^k \frac{\partial}{\partial x^i}|_x,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x \in T_xM$. The coefficients ${R^i}_k = {R^i}_k(x, y)$ are given by

$${R^i}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{9}$$

It follows from (6) that

$$\mathbf{R}_y(y) = 0, \qquad \mathbf{g}_y(\mathbf{R}_y(u), v) = \mathbf{g}_y(u, \mathbf{R}_y(v)), \tag{10}$$

where $u, v \in T_xM$. The family $\mathbf{R} := \{\mathbf{R}_y | y \in T_xM \setminus \{0\}\}$ is called the *Riemann curvature*.

Using the Chern connection ∇ on π^*TM, one can define the covariant derivative of a vector field $X = X^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ along a curve c by

$$D_{\dot c}X(t) := \Big\{ \frac{dX^i}{dt}(t) + X^j(t)\Gamma^i_{jk}(c(t), \dot c(t))\dot c^k(t) \Big\} \frac{\partial}{\partial x^i}|_{c(t)}.$$

If $H = H(s,t)$ is a family of geodesics, i.e., for each s, $\sigma_s(t) := H(s,t)$ is a geodesic, the variation field $V_s(t) := \frac{\partial H}{\partial s}(s,t)$ satisfies the following Jacobi field along σ_s,

$$D_{\dot\sigma_s} D_{\dot\sigma_s} V_s(t) + \mathbf{R}_{\dot\sigma_s(t)}(V(t)) = 0.$$

For a tangent plane $P \subset T_xM$ and a vector $0 \neq y \in P$, let

$$\mathbf{K}(P,y) := \frac{\mathbf{g}_y(\mathbf{R}_y(u),u))}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - [\mathbf{g}_y(y,u)]^2},$$

where $P = \mathrm{span}\{y,u\}$. By (10), one can see that $\mathbf{K}(P,y)$ is well-defined, namely, independent of the choice of a particular $u \in T_xM$.

The mean Cartan tensor and the mean Landsberg tensor can be viewed as families of vectors on the manifold, i.e.,

$$\mathbf{I}_y = I^i(x,y)\frac{\partial}{\partial x^i}|_x, \qquad \mathbf{J}_y = J^i(x,y)\frac{\partial}{\partial x^i}|_x,$$

where $I^i := g^{il}I_l$ and $J^i := g^{il}J_l$. It follows from (4) and (8) that

$$\mathbf{g}_y(\mathbf{I}_y, y) = 0 = \mathbf{g}_y(\mathbf{J}_y, y).$$

Thus $\mathbf{I}_y$ and $\mathbf{J}_y$ are perpendicular to y with respect to $\mathbf{g}_y$. We call $\mathbf{I} := \{\mathbf{I}_y \mid y \in TM \setminus \{0\}\}$ and $\mathbf{J} := \{\mathbf{J}_y \mid y \in TM \setminus \{0\}\}$ the *mean Cartan torsion* and the *Landsberg curvature*, respectively.

If F is a Berwald metric, then $\mathbf{J} = 0$ and $\mathbf{S} = 0$. The converse is true too in dimension two, but it is not clear in higher dimensions (Cf. [17]).

3 Finsler metrics with constant S-curvature

The following lemma is cruial for the proof of Theorem 1.1.

Lemma 3.1 *Let (M,F) be an n-dimensional Finsler manifold. Suppose that there is a constant c and a closed 1-form γ such that*

$$\mathbf{S}(x,y) = (n+1)cF(x,y) + \gamma_x(y), \qquad y \in T_xM,$$

then along any geodesic $\sigma = \sigma(t)$, the vector field $\mathbf{I}(t) := I^i(\sigma(t),\dot\sigma(t))\frac{\partial}{\partial x^i}|_{\sigma(t)}$ satisfies the following equation:

$$\mathrm{D}_{\dot\sigma}\mathrm{D}_{\dot\sigma}\mathbf{I}(t) + \mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)) = 0. \tag{11}$$

Proof: It is known that the Landsberg tensor satisfies the following equation [13] [15] :

$$J_{k|m}y^m + I_m R^m_{\ k} = -\frac{1}{3}\Big\{2R^m_{\ k\cdot m} + R^m_{\ m\cdot k}\Big\} \tag{12}$$

and the S-curvature satisfies the following equation [8] [14]:

$$\mathbf{S}_{\cdot k|m} y^m - \mathbf{S}_{|k} = -\frac{1}{3}\Big\{ 2R^m{}_{k\cdot m} + R^m{}_{m\cdot k}\Big\}. \tag{13}$$

It follows from (12) and (13) that

$$J_{k|m} y^m + I_m R^m_{\ k} = \mathbf{S}_{\cdot k|m} y^m - \mathbf{S}_{|k}.$$

By (7), we can rewrite the above equation as follows

$$I^i{}_{|p|q} y^p y^q + R^i{}_m I^m = g^{ik}\Big\{ \mathbf{S}_{\cdot k|m} y^m - \mathbf{S}_{|k}\Big\}. \tag{14}$$

Note that $F = \sqrt{g_{ij} y^i y^j}$ satisfies

$$F_{|m} = \frac{g_{ij|m} y^i y^j}{2F} = 0, \qquad F_{\cdot k|m} = \frac{g_{ik|m} y^i}{F} = 0.$$

Since $\gamma = \gamma_i dx^i$ is closed, it satisfies

$$\gamma_{\cdot k|m} y^m - \gamma_{|k} = \Big\{ \frac{\partial \gamma_k}{\partial x^m} - \frac{\partial \gamma_m}{\partial x^k}\Big\} y^m = 0.$$

We have

$$\mathbf{S}_{\cdot k|m} y^m - \mathbf{S}_{|k} = (n+1)c\Big\{ F_{\cdot k|m} y^m - F_{|k}\Big\} + \gamma_{\cdot k|m} y^m - \gamma_{|k} = 0.$$

Then (14) is reduced to

$$I^i{}_{|p|q} y^p y^q + R^i{}_m I^m = 0. \tag{15}$$

Since σ is a geodesic, we have

$$D_{\dot\sigma} D_{\dot\sigma} \mathbf{I}(t) = I^i{}_{|p|q}(\sigma(t), \dot\sigma(t)) \dot\sigma^p(t) \dot\sigma^q(t) \frac{\partial}{\partial x^i}|_{\sigma(t)}.$$

Then (15) restricted to $\sigma(t)$ gives rise to (11). Q.E.D.

4 Proof of Theorem 1.1

In this section, we are going to prove a slightly more general version of Theorem 1.1.

Theorem 4.1 *Let (M, F) be an n-dimensional complete Finsler manifold with nonpositive flag curvature $\mathbf{K} \le 0$ and almost constant S-curvature $\mathbf{S} = (n+1)cF + \gamma$ (c = constant and γ is a closed 1-form). Suppose that the mean Cartan torsion grows sub-linearly. Then $\mathbf{J} = 0$ and $\mathbf{K}(P, y) = 0$ for the flag $P = \mathrm{span}\{\mathbf{I}_y, y\}$ whenever $\mathbf{I}_y \ne 0$. Moreover F is Riemannian at points where $\mathbf{K} < 0$.*

Proof: Let $y \in T_xM$ be an arbitrary non-zero vector and let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Since the Finsler metric is complete, one may assume that σ is defined on $(-\infty, \infty)$. $\mathbf{I}$ and $\mathbf{J}$ restricted to σ are vector fields along σ,

$$\mathbf{I}(t) := I^i\Big(\sigma(t), \dot\sigma(t)\Big)\frac{\partial}{\partial x^i}|_{\sigma(t)}, \qquad \mathbf{J}(t) := J^i\Big(\sigma(t), \dot\sigma(t)\Big)\frac{\partial}{\partial x^i}|_{\sigma(t)}.$$

It follows from (7) that

$$D_{\dot\sigma}\mathbf{I}(t) = I^i{}_{|m}\Big(\sigma(t), \dot\sigma(t)\Big)\dot\sigma^m(t)\frac{\partial}{\partial x^i}|_{\sigma(t)} = \mathbf{J}(t). \tag{16}$$

If $\mathbf{I}(t) \equiv 0$, then by (16), $\mathbf{J}_y = D_{\dot\sigma}\mathbf{I}(0) = 0$. From now on, we assume that $\mathbf{I}(t) \not\equiv 0$. Let

$$\varphi(t) := \sqrt{\mathbf{g}_{\dot\sigma(t)}\Big(\mathbf{I}(t), \mathbf{I}(t)\Big)}. \tag{17}$$

Let $I = (a, b) \neq \emptyset$ be a maximal interval on which $\varphi(t) > 0$. We have

$$\varphi\varphi' = \mathbf{g}_{\dot\sigma}\Big(\mathbf{I}, D_{\dot\sigma}\mathbf{I}\Big) \le \sqrt{\mathbf{g}_{\dot\sigma}\Big(\mathbf{I}, \mathbf{I}\Big)}\sqrt{\mathbf{g}_{\dot\sigma}\Big(D_{\dot\sigma}\mathbf{I}, D_{\dot\sigma}\mathbf{I}\Big)} = \varphi\sqrt{\mathbf{g}_{\dot\sigma}\Big(D_{\dot\sigma}\mathbf{I}, D_{\dot\sigma}\mathbf{I}\Big)}.$$

This is,

$$\varphi' \le \sqrt{\mathbf{g}_{\dot\sigma}\Big(D_{\dot\sigma}\mathbf{I}, D_{\dot\sigma}\mathbf{I}\Big)}. \tag{18}$$

By assumption $\mathbf{K} \le 0$ and (18), we have

$$\begin{aligned}\frac{1}{2}[\varphi^2]'' &= \mathbf{g}_{\dot\sigma}\Big(\mathrm{D}_{\dot\sigma}\mathrm{D}_{\dot\sigma}\mathbf{I}, \mathbf{I}\Big) + \mathbf{g}_{\dot\sigma}\Big(\mathrm{D}_{\dot\sigma}\mathbf{I}, \mathrm{D}_{\dot\sigma}\mathbf{I}\Big)\\ &= -\mathbf{g}_{\dot\sigma}\Big(\mathbf{R}_{\dot\sigma}(\mathbf{I}), \mathbf{I}\Big) + \mathbf{g}_{\dot\sigma}\Big(\mathrm{D}_{\dot\sigma}\mathbf{I}, \mathrm{D}_{\dot\sigma}\mathbf{I}\Big)\\ &\ge \mathbf{g}_{\dot\sigma}\Big(\mathrm{D}_{\dot\sigma}\mathbf{I}, \mathrm{D}_{\dot\sigma}\mathbf{I}\Big) \ge \varphi'^2.\end{aligned} \tag{19}$$

We obtain that $\varphi''(t) \ge 0$.

We claim that $\varphi'(t) \equiv 0$. Suppose that $\varphi'(t_o) \neq 0$ for some $t_o \in I$. If $\varphi'(t_o) > 0$, then

$$\varphi(t) \ge \varphi'(t_o)(t - t_o) + \varphi(t_o), \qquad t > t_o.$$

Thus $b = +\infty$. If $\varphi'(t_o) < 0$, then

$$\varphi(t) \ge \varphi'(t_o)(t - t_o) + \varphi(t_o) > \varphi(t_o) > 0, \qquad t < t_o.$$

Thus $a = -\infty$. In either case, $\varphi(t)$ grows at least linearly. Note that for $p = \sigma(t_o)$, $\mathcal{I}_p(|t - t_o|) \ge \varphi(t)$. We see that $\mathbf{I}$ grows at least linearly. This is impossible. Thus $\varphi'(t) \equiv 0$ and $\varphi(t) = \mathit{constant} > 0$. In this case, $I = (-\infty, \infty)$.

It follows from (19) that

$$\mathbf{g}_{\dot\sigma}\Big(\mathbf{R}_{\dot\sigma}(\mathbf{I}), \mathbf{I}\Big) = 0, \qquad \mathrm{D}_{\dot\sigma}\mathbf{I} = 0.$$

By (16), we get $\mathbf{J}_y = D_{\dot{\sigma}}\mathbf{I}(0) = 0$. Since $\mathbf{I}_y$ is orthogonal to y with respect to $\mathbf{g}_y$, $\mathbf{K}(P, y) = 0$ for $P = \mathrm{span}\{\mathbf{I}_y, y\}$ whenever $\mathbf{I}_y \neq 0$.

Assume that $\mathbf{K} < 0$ at a point $x \in M$. It follows from $\mathbf{g}_y(\mathbf{R}_y(\mathbf{I}_y), \mathbf{I}_y) = 0$ that $\mathbf{I}_y = 0$ for all $y \in T_xM \setminus \{0\}$. By Deicke's theorem [11], F is Riemannian. Q.E.D.

Two natural problems arise:

(a) Is there any *complete* non-Landsberg metric on R^n with $\mathbf{K} \leq 0$, $\mathbf{S} = (n+1)cF$ and $\mathcal{I}_p(r) \sim Cr$ (as $r \to +\infty$)?

(b) What is the metric structure of a complete Finsler metric on R^n $(n \geq 3)$ satisfying $\mathbf{K} = 0$, $\mathbf{J} = 0$ and $\mathbf{S} = 0$?

If the flag curvature is strictly negative, we have the following

Theorem 4.2 *Let (M, F) be an n-dimensional complete Finsler manifold with $\mathbf{K} \leq -1$ and almost constant S-curvature. Suppose that the mean Cartan torsion $\mathbf{I}$ grows sub-exponentially a rate of $k = 1$. Then F is Riemannian.*

Proof: The proof is similar. Assume that $\mathbf{I}_y \neq 0$ for some non-zero vector $y \in T_xM$. Let $y \in T_xM$ be an arbitrary vector and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Let $\varphi(t)$ be defined by (17). Let $I = (a, b) \neq \emptyset$ be the maximal interval on which $\varphi(t) > 0$ and $0 \in I$. By assumption $\mathbf{K} \leq -1$ and (18), we obtain

$$\begin{aligned} \frac{1}{2}[\varphi^2]'' &= \mathbf{g}_{\dot{\sigma}}\Big(\mathrm{D}_{\dot{\sigma}}\mathrm{D}_{\dot{\sigma}}\mathbf{I}, \mathbf{I}\Big) + \mathbf{g}_{\dot{\sigma}}\Big(\mathrm{D}_{\dot{\sigma}}\mathbf{I}, \mathrm{D}_{\dot{\sigma}}\mathbf{I}\Big) \\ &= -\mathbf{g}_{\dot{\sigma}}\Big(\mathbf{R}_{\dot{\sigma}}(\mathbf{I}), \mathbf{I}\Big) + \mathbf{g}_{\dot{\sigma}}\Big(\mathrm{D}_{\dot{\sigma}}\mathbf{I}, \mathrm{D}_{\dot{\sigma}}\mathbf{I}\Big) \\ &\geq \varphi^2 + \varphi'^2. \end{aligned}$$

This gives rise to the following inequality

$$\varphi'' - \varphi \geq 0. \tag{20}$$

We claim that $\varphi'(t) \equiv 0$. Suppose that $\varphi'(t_o) \neq 0$ for some $t_o \in I$. Let

$$\varphi_o(t) := \varphi(t_o)\cosh(t - t_o) + \varphi'(t_o)\sinh(t - t_o).$$

Let $h(t) := \varphi'(t)/\varphi(t)$ and $h_o(t) := \varphi_o'(t)/\varphi_o(t)$.

$$\chi(t) := e^{\int_{t_o}^{t}(h(\tau)+h_o(\tau))d\tau}\Big[h(t) - h_o(t)\Big].$$

It is easy to verify that $\chi'(t) \geq 0$ and $\chi(t_o) = 0$. Thus $\chi(t) \geq 0$ for $t > t_o$ and $\chi(t) \leq 0$ for $t < t_o$. This implies that $h(t) \geq h_o(t)$ for $t > t_o$ and $h(t) \leq h_o(t)$ for $t < t_o$. Note that

$$h(t) - h_o(t) = \frac{\varphi'(t)}{\varphi(t)} - \frac{\varphi_o'(t)}{\varphi_o(t)} = \frac{d}{dt}\Big[\ln\frac{\varphi(t)}{\varphi_o(t)}\Big].$$

Thus $[\varphi/\varphi_o]'(t) \geq 0$ for $t > t_o$ and $[\varphi/\varphi_o]'(t) \leq 0$ for $t < t_o$. We conclude that

$$\varphi(t) \geq \varphi_o(t), \qquad a < t < b.$$

If $\varphi'(t_o) > 0$, then

$$\varphi(t) \geq \varphi_o(t) > 0, \qquad t > t_o.$$

Thus $b = +\infty$ and

$$\liminf_{t\to+\infty} \frac{\varphi(t)}{e^{t-t_o}} \geq \frac{\varphi(t_o) + \varphi'(t_o)}{2} > 0.$$

If $\varphi'(t_o) < 0$, then

$$\varphi(t) \geq \varphi_o(t) > 0, \qquad t < t_o.$$

Thus $a = -\infty$ and

$$\liminf_{t\to-\infty} \frac{\varphi(t)}{e^{t-t_o}} \geq \frac{\varphi(t_o) - \varphi'(t_o)}{2} > 0.$$

Note that $\mathcal{I}_p(|t - t_o|) \geq \varphi(t)$ for $p = \sigma(t_o)$. Thus $\mathbf{I}$ grows at least exponentially at rate of $k = 1$. This contradicts the assumption. Therefore, $\varphi'(t) \equiv 0$.

Since $\varphi'(t) \equiv 0$, we conclude that $\varphi(t) \equiv 0$ by (20). In particular, $\mathbf{I}_y = \varphi(0) = 0$. This contradicts the assumption at the beginning of the argument.

Therefore $\mathbf{I} \equiv 0$ and F is Riemannian by Deicke's theorem [11]. Q.E.D.

A natural problem arises: Is there any non-Riemannian *complete* Finsler metric on R^n satisfying $\mathbf{K} \leq -1$, $\mathbf{S} = (n+1)cF$, but $\mathcal{I}_p(r) \sim Ce^r$ (as $r \to +\infty$)? This problems remains open.

Example 4.3 Let $\phi = \phi(y)$ be a Minkowski norm on R^n and $\mathcal{U} := \{y \in \mathrm{R}^n \mid \phi(y) < 1\}$. Let $\Theta = \Theta(x, y)$ be a function on $T\mathcal{U} \cong \mathcal{U} \times \mathrm{R}^n$ defined by

$$\Theta(x, y) = \phi\Big(y - \Theta(x, y)x\Big).$$

Θ is a Finsler metric on $\mathcal{U}$ which is called the *Funk metric* [12]. The Funk metric satisfies the following important equation

$$\Theta_{x^k}(x, y) = \Theta(x, y)\Theta_{y^k}(x, y). \tag{21}$$

Let $a \in \mathrm{R}^n$ be an arbitrary constant vector $a \in \mathrm{R}^n$ with $|a| < 1$. Let

$$F := \Theta(x, y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \qquad y \in T\mathcal{U} \cong \mathcal{U} \times \mathrm{R}^n.$$

Clearly, F is a Finsler metric near the origin. By (21), one sees that the spray coefficients of F are given by $G^i = Py^i$, where

$$P := \frac{1}{2}\Big\{\Theta(x, y) - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\Big\}.$$

Then using the above formula for G^i and (9), one can easily show that F has constant flag curvature $\mathbf{K} = -\frac{1}{4}$ (see Example 5.3 in [19]). Now let us compute the S-curvature of F. A direct computation gives

$$\frac{\partial G^m}{\partial y^m} = (n+1)P.$$

Let $dV = \sigma_F(x)dx^1 \cdots dx^n$ be the Finsler volume form on M. Using (5), we obtain

$$\begin{aligned}
\mathbf{S} &= (n+1)P(x,y) - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F(x)\Big) \\
&= \frac{n+1}{2}F(x,y) - (n+1)\frac{\langle a, y\rangle}{1+\langle a, x\rangle} - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F(x)\Big) \\
&= \frac{1}{2}(n+1)F(x,y) + d\varphi_x(y),
\end{aligned}$$

where

$$\varphi := -\ln\Big[(1+\langle a, x\rangle)^{n+1}\sigma_F(x)^{\frac{1}{n+1}}\Big]. \tag{22}$$

Thus F has almost constant S-curvature. Note that F is not Riemannian in general.

Example 4.3 shows that the completeness in Theorem 4.1 can not be replaced by the positive completeness.

5 An Example

The local/global structures of Berwald metrics have been completely determined by Z.I. Szabo [21], but their curvature properties have not been discussed throughly. Here we are going to compute the Riemann curvature and the mean Cartan torsion for a special class of Berwald manifolds constructed from a pair of Riemannian manifolds. Then we show that these metrics satisfy the conditions and the conclusions in Theorem 1.1 (b).

Let (M_i, α_i), $i = 1, 2$, be arbitrary Riemannian manifolds and $M = M_1 \times M_2$. Let $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ be an arbitrary C^∞ function satisfying

$$f(\lambda s, \lambda t) = \lambda f(s,t), \quad (\lambda > 0) \quad \text{and} \quad f(s,t) \neq 0 \text{ if } (s,t) \neq 0.$$

Define

$$F := \sqrt{f\Big([\alpha_1(x_1,y_1)]^2,\ [\alpha_2(x_2,y_2)]^2\Big)}, \tag{23}$$

where $x = (x_1, x_2) \in M$ and $y = y_1 \oplus y_2 \in T_{(x_1,x_2)}(M_1 \times M_2) \cong T_{x_1}M_1 \oplus T_{x_2}M_2$. Clearly, F has the following properties:

(a) $F(x,y) \geq 0$ with equality holds if and only if $y = 0$;

(b) $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$;

(c) $F(x, y)$ is C^∞ on $TM \setminus \{0\}$.

Now we are going to find additional condition on $f = f(s,t)$ under which the matrix $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ is positive definite. Take standard local coordinate systems (x^a, y^a) in TM_1 and (x^α, y^α) in TM_2. Then $(x^i, y^j) := (x^a, x^\alpha, y^a, y^\alpha)$ is a standard local coordinate system in TM. Express

$$\alpha_1(x_1, y_1) = \sqrt{\bar{g}_{ab}(x_1)y^a y^b}, \qquad \alpha_2(x_2, y_2) = \sqrt{\bar{g}_{\alpha\beta}(x_2)y^\alpha y^\beta},$$

where $y_1 = y^a \frac{\partial}{\partial x^a}$ and $y_2 = y^\alpha \frac{\partial}{\partial x^\alpha}$. We obtain

$$\Big(g_{ij}\Big) = \begin{pmatrix} 2f_{ss}\bar{y}_a\bar{y}_b + f_s\bar{g}_{ab} & 2f_{st}\bar{y}_a\bar{y}_\beta \\ 2f_{st}\bar{y}_b\bar{y}_\alpha & 2f_{tt}\bar{y}_\alpha\bar{y}_\beta + f_t\bar{g}_{\alpha\beta} \end{pmatrix}, \tag{24}$$

where $\bar{y}_a := \bar{g}_{ab}y^b$ and $\bar{y}_\alpha := \bar{g}_{\alpha\beta}y^\beta$. By an elementary argument, one can show that $\Big(g_{ij}\Big)$ is positive definite if and only if f satisfies the following conditions:

$$f_s > 0, \qquad f_t > 0, \qquad f_s + 2sf_{ss} > 0, \qquad f_t + 2tf_{tt} > 0,$$

and

$$f_s f_t - 2ff_{st} > 0.$$

In this case,

$$\det\Big(g_{ij}\Big) = h\Big([\alpha_1]^2,\ [\alpha_2]^2\Big)\det\Big(\bar{g}_{ab}\Big)\det\Big(\bar{g}_{\alpha\beta}\Big), \tag{25}$$

where

$$h := (f_s)^{n_1-1}(f_t)^{n_2-1}\Big\{f_sf_t - 2ff_{st}\Big\},$$

where $n_1 := \dim M_1$ and $n_2 := \dim M_2$.

By a direct computation, one knows that the spray coefficients of F are splitted as the direct sum of the spray coefficients of α_1 and α_2, that is,

$$G^a(x, y) = \bar{G}^a(x_1, y_1), \qquad G^\alpha(x, y) = \bar{G}^\alpha(x_1, y_1), \tag{26}$$

where $\bar{G}^a$ and $\bar{G}^\alpha$ are the spray coefficients of α_1 and α_2 respectively. From (26), one can see that the spray of F is independent of the choice of a particular function f. In particular, G^i are quadratic in $y \in T_xM$. Thus F is a Berwald metric. This fact is claimed in [21]. Since F is a Berwald metric, $\mathbf{J} = 0$ and $\mathbf{S} = 0$ [17].

The Riemann tensor of F is given by

$$\Big(R^i{}_j\Big) = \Big(\bar{R}^i{}_j\Big) = \begin{pmatrix} \bar{R}^a{}_b & 0 \\ 0 & \bar{R}^\alpha{}_\beta \end{pmatrix},$$

where $\bar{R}^a{}_b$ and $\bar{R}^\alpha{}_\beta$ are the coefficients of the Riemann tensor of α_1 and α_2 respectively. Let $R_{ij} := g_{ik}R^k{}_j$, $\bar{R}_{ab} := \bar{g}_{ac}\bar{R}^c{}_b$ and $\bar{R}_{\alpha\beta} := \bar{g}_{\alpha\gamma}\bar{R}^\gamma{}_\beta$. Using (24), one obtains

$$\Big(R_{ij}\Big) = \begin{pmatrix} f_s\bar{R}_{ab} & 0 \\ 0 & f_t\bar{R}_{\alpha\beta} \end{pmatrix}.$$

For any vector $v = v^i \frac{\partial}{\partial x^i}|_x \in T_xM$,

$$\mathbf{g}_y\Big(\mathbf{R}_y(v), v\Big) = f_s\bar{R}_{ab}v^a v^b + f_t\bar{R}_{\alpha\beta}v^\alpha v^\beta. \tag{27}$$

It follows from (27) that if α_1 and α_2 both have nonpositive sectional curvature, then F has nonpositive flag curvature.

Using (25), one can compute the mean Cartan torsion. First, observe that

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{jk})}\Big] = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{h\Big([\alpha_1]^2, [\alpha_2]^2\Big)}\Big].$$

One obtains

$$I_a = \frac{h_s}{h}\bar{y}_a \qquad I_\alpha = \frac{h_t}{h}\bar{y}_\alpha,$$

where $\bar{y}_a := \bar{g}_{ab}y^b$ and $\bar{y}_\alpha := \bar{g}_{\alpha\beta}y^\beta$. Since $\bar{y}_a\bar{R}^a{}_b = 0$ and $\bar{y}_\alpha\bar{R}^\alpha{}_\beta = 0$, one obtains

$$\mathbf{g}_y\Big(\mathbf{R}_y(\mathbf{I}_y), \mathbf{I}_y\Big) = I_iR^i{}_jI^j = \frac{h_s}{h}\bar{y}_a\bar{R}^a{}_bI^b + \frac{h_t}{h}\bar{y}_\alpha\bar{R}^\alpha{}_\beta I^\beta = 0.$$

Therefore F satisfies the conditions and conclusions in Theorem 4.1.

References

[1] H. Akbar-Zadeh, *Sur les espaces de Finsler á courbures sectionnelles constantes*, Bull. Acad. Roy. Bel. Cl, Sci, 5e Série - Tome LXXXIV (1988), 281-322.

[2] D. Bao, S. S. Chern and Z. Shen, *Rigidity issues on Finsler surfaces*, Rev. Roumaine Math. Pures Appl. **42**(1997), 707-735.

[3] D. Bao and C. Robles, *On Randers metrics of constant curvature*, Reports on Mathematical Physics, (to appear).

[4] D. Bao, C. Robles and Z. Shen, *Zermelo Navigation on Riemannian manifolds*, preprint.

[5] D. Bao and Z. Shen, *Finsler metrics of constant curvature on the Lie group* S^3, J. of the London Math. Soc. **66**(2002), 453-467.

[6] L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z. **25**(1926), 40-73.

[7] L. Berwald, *Parallelübertragung in allgemeinen Räumen*, Atti Congr. Intern. Mat. Bologna **4**(1928), 263-270.

[8] X. Chen, X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, J. of the London Math. Soc., (to appear).

[9] X. Chen and Z. Shen, *Randers metrics with special curvature properties*, Osaka J. of Math. **40**(2003), 87-101.

[10] S. S. Chern, *On the Euclidean connections in a Finsler space*, Proc. National Acad. Soc., **29**(1943), 33-37; or Selected Papers, vol. II, 107-111, Springer 1989.

[11] A. Deicke, *Über die Finsler-Raume mit* $A_i = 0$, Arch. Math. **4**(1953), 45-51.

[12] P. Funk, *Über Geometrien, bei denen die Geraden die Kürzesten sind*, Math. Annalen **101**(1929), 226-237.

[13] X. Mo, *The flag curvature tensor on a closed Finsler space*, Results in Math. **36**(1999), 149-159.

[14] X. Mo, *On the flag curvature of a Finsler space with constant S-curvature*, Houston J. of Math. (to appear).

[15] X. Mo and Z. Shen, *On negatively curved Finsler manifolds of scalar curvature*, Canadian Mathematical Bulletin, to appear.

[16] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Math. **128**(1997), 306-328.

[17] Z. Shen, *Lectures on Finsler Geometry*, World Scientific, Singapore (2001), 307 pages.

[18] Z. Shen, *Projectively flat Randers metrics with constant flag curvature*, Math. Ann. **325**(2003), 19-30.

[19] Z. Shen, *Projectively flat Finsler metrics of constant flag curvature*, Trans. of Amer. Math. Soc. **355**(4)(2003), 1713-1728.

[20] Z. Shen, *Finsler metrics with* $\mathbf{K} = 0$ *and* $\mathbf{S} = 0$, Canadian J. Math. **55**(2003), 112-132.

[21] Z.I. Szabó, *Positive definite Berwald spaces (Structure theorems on Berwald spaces)*, Tensor, N. S. **35**(1981), 25–39.

Zhongmin Shen
Dept of Math, IUPUI
402 N. Blackford Street
Indianapolis, IN 46202-3216, USA.
zshen@math.iupui.edu
www.math.iupui.edu/~zshen